

# On infinite effectivity of motivic spectra and the vanishing of their motives

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December 6, 2016

## Abstract

We study the kernel of the "compact motivization" functor  $M_{k,\Lambda}^c : SH_{\Lambda}^c(k) \rightarrow DM_{\Lambda}^c(k)$  (i.e., we try to describe those compact objects of the  $\Lambda$ -linear version of  $SH(k)$  whose associated motives vanish; here  $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$ ). We also investigate the question when the  $m$ -homotopy connectivity of  $M_{k,\Lambda}^c(E)$  ensures the  $m$ -homotopy connectivity of  $E$  itself (with respect to the homotopy  $t$ -structure  $t_{\Lambda}^{SH}$  for  $SH_{\Lambda}(k)$ ). We prove that the kernel of  $M_{k,\Lambda}^c$  vanishes and the corresponding "homotopy connectivity detection" statement is also valid if and only if  $k$  is a non-orderable field; this is an easy consequence of similar results of T. Bachmann (who considered the case where the cohomological 2-dimension of  $k$  is finite). We also sketch a deduction of these statements from the "slice-convergence" results of M. Levine. Moreover, for an arbitrary  $k$  the kernel in question does not contain any 2-torsion (and the author also suspects that all its elements are odd torsion unless  $\frac{1}{2} \in \Lambda$ ). Furthermore, if the exponential characteristic of  $k$  is inverted in  $\Lambda$  then this kernel consists exactly of "infinitely effective" (in the sense of Voevodsky's slice filtration) objects of  $SH_{\Lambda}^c(k)$ .

The author believes that the results of this paper will become a useful tool for the study of motivic spectra. In particular, they enable us (following another idea of Bachmann) to carry over his results on the tensor invertibility of certain motives of affine quadrics to the corresponding motivic spectra whenever  $k$  is non-orderable. We also generalize a theorem of A. Asok.

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\*Research is supported by the Russian Science Foundation grant no. 16-11-10200 .

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# Introduction

It is well known that for a (perfect) field  $k$  both the (Morel-Voevodsky's) motivic stable homotopy category  $SH(k)$  and Voevodsky's motivic category  $DM(k)$  are important for the study of cohomology of  $k$ -varieties. The roles of these categories are somewhat distinct: whereas  $SH(k)$  is "closer to the geometry" of varieties,  $DM(k)$  is somewhat easier to deal with. For instance, we know much more on morphisms in  $DM(k)$  than in  $SH(k)$ ; this information yields the existence of the so-called Chow weight structure for  $DM^c(k) \subset DM(k)$  (as shown in [Bon10] and [Bon11]; below we will mention an interesting application of this result described in [Bach15]).

Now, there is a connecting functor  $M_k : SH(k) \rightarrow DM(k)$  (that sends the motivic spectra of smooth varieties into their motives); so it is rather important to describe the extent to which  $M_k$  is conservative. Whereas the "whole"  $M_k$  is never conservative (see Remark 2.1.2(2) below), (the "compact

version" (b) of) Theorem 15 of [Bach16] states that the restriction  $M_k^c$  of  $M_k$  to compact objects is conservative whenever  $k$  is of finite cohomological 2-dimension.

The current paper grew out of the following observation: this theorem can be generalized to the case of an arbitrary non-orderable (perfect)  $k$  via a simple "continuity" argument (i.e., if  $k = \varinjlim k_i$  then the conservativity of all  $M_{k_i}^c$  implies that for  $M_k^c$ ). We also note (in Remark 2.1.2(3)) that this conservativity statement fails whenever  $k$  is a formally real field (though we conjecture that the kernel of  $M_k^c$  consists of torsion elements only; see Remark 2.1.2(4)). Moreover, we extend to arbitrary non-orderable fields the stronger part of Bachmann's Theorem 15(b); so we prove that the  $m$ -homotopy connectivity of  $M_k^c(E)$  (for  $E \in \text{Obj } SH^c(k)$ ) ensures the  $m$ -homotopy connectivity of  $E$  itself (here  $m$ -homotopy connectivity means belonging to the  $t_{\text{hom}} \geq m + 1$ -part for the corresponding homotopy  $t$ -structure). Lemma 18 of *ibid.* also yields a similar result for  $E$  being a 2-torsion compact motivic spectrum over an arbitrary perfect  $k$  (note that the formulation of the latter statement in the first version of this paper has motivated Bachmann to prove *loc. cit.*; in his lemma the case where  $k$  is of finite virtual cohomological 2-dimension is considered).

Though these continuity arguments are rather simple, the author believes that the results described above are quite useful. To illustrate their utility, we (easily) deduce a certain generalization of Theorem 2.2.1 of [Aso16]. So, we extend this theorem (in Proposition 2.3.4) to the case of an arbitrary non-orderable perfect base field (and to a not necessarily proper  $X/k$ ). Besides (for the sake of generality; in this proposition as well as in the central results of this paper) we actually consider the  $\Lambda$ -linear versions of the statements described above, where  $\Lambda$  is an arbitrary (unital commutative) coefficient subring of  $\mathbb{Q}$ . For a smooth proper  $X$  this corresponds to studying the conditions ensuring that  $X$  contains a 0-cycle whose degree is invertible in  $\Lambda$  and that the kernel of the degree homomorphism  $\text{Chow}_0(X_L) \rightarrow \mathbb{Z}$  is killed by  $-\otimes_{\mathbb{Z}} \Lambda$  for any field extension  $L/k$  (see Remark 2.3.5(2)); so, the case  $\Lambda \neq \mathbb{Z}$  may be quite interesting (also). Under the condition that  $p$  is invertible in  $\Lambda$  whenever it is positive, this equivalence result may be vastly generalised (using the results of [BoS14] and *Chow-weight homology*); see Remark 2.3.5(3).

This  $\Lambda$ -linear setting has some more advantages. In particular, we describe an argument deducing our central Theorem 2.3.1(i) from the "slice-convergence" results of [Lev13] (avoiding the usage of [Bach16]); yet for this

argument we have to assume that the characteristic  $p$  of  $k$  is invertible in  $\Lambda$  whenever it is positive. The basic idea of both versions of the proof is that "compact" statements can often be reduced to the case of a "small" base field (using the continuity of  $SH_{\Lambda}^c(-)$  in the sense described in §1.1); this allows to deduce them from the (already known) "slice-convergence" properties of objects of certain subcategories of  $SH_{\Lambda}(k)$  larger than  $SH_{\Lambda}^c(k)$ ; cf. Remark 3.2.1(1) below.

Another application of our results (that also generalizes another statement formulated by Bachmann and requires a coefficient ring distinct from  $\mathbb{Z}$  if the characteristic of  $k$  is positive) is the following one: the so-called unreduced suspension (as defined in [Poh05]) of  $\Sigma_{T,\Lambda}^{\infty}(A_+)$  is  $\otimes$ -invertible in  $SH_{\Lambda}$  whenever  $k$  is non-orderable,  $A$  is the (affine) zero set of  $\phi - a$  for  $\phi$  being a non-zero quadratic form,  $0 \neq a \in k$ ,  $p$  is distinct from 2 and is invertible in  $\Lambda$  if it is positive. We deduce this statement from Theorem 30 of [Bach15] (whose proof is based on the usage of the Chow weight structure for  $DM^c(k)[\frac{1}{p}] \subset DM(k)[\frac{1}{p}]$ ).

Now we describe the "most original" result of this paper (at least, it appears not to be formulated in the literature in any form). We prove that an object  $E$  of  $SH_{\Lambda}^c(k)$  belongs to  $SH_{\Lambda}^{eff}(k)(r)$  (to the  $r$ th level of the  $\Lambda$ -linearized version of the Voevodsky's slice filtration; we also say that the objects of  $SH_{\Lambda}^{eff}(k)(r)$  are  $r$ -effective) if and only if  $M_{k,\Lambda}(E)$  belongs to  $DM_{\Lambda}^{eff}(k)(r)$ . We also establish a certain " $t_{hom}$ -connective" version of this statement. Assuming that  $p$  is invertible in  $\Lambda$  whenever it is positive, we immediately deduce the "infinite effectivity" of the compact motivization kernel, and describe when  $M_{k,\Lambda}(E) \in DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq m+1}$  for  $E \in SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq m}$ .

Let us now describe the contents of the paper. Some more information of this sort can be found at the beginnings of sections.

In §1 we recall some basics on (general) triangulated categories,  $SH(-)$  and  $DM(-)$ , on the cohomological dimension of fields and their Grothendieck-Witt rings of quadratic forms. We also introduce the  $\Lambda$ -linear versions of  $SH(-)$  and  $DM(-)$  and discuss certain continuity arguments.

In §2 we recall some more results on motivic categories (we formulate them in the  $\Lambda$ -linear setting). They enable us to generalize several results of Bachmann (as well as Theorem 2.2.1 of [Aso16]) to the case of arbitrary non-orderable base fields. We also prove that the restriction of  $M_{k,\Lambda}^c$  onto 2-torsion objects is conservative over any perfect  $k$ ; we conjecture that this kernel consists of odd torsion objects only whenever  $\frac{1}{2} \notin \Lambda$ . We also note

that the Morel's morphism  $\eta$  is torsion if and only if  $k$  is non-orderable; this is also equivalent to the vanishing of the  $SH^-(k)$ -parts in the Morel's decompositions of  $SH[\frac{1}{2}](k)$  and  $SH_{\mathbb{Q}}(k)$ .

§3 we prove that the compact motivization functor  $M_{k,\Lambda}^c$  "strictly respects" the slice filtrations (on  $SH_{\Lambda}^c(k)$  and  $DM_{\Lambda}^c(k)$ , respectively) as well as the (more precise) homotopy  $t$ -structure analogue of this result. These statements yield an alternative method for proving Theorem 2.3.1(i) (that is the central result of this paper); so we sketch an argument deducing it from the results of [Lev13] (under the additional condition that  $p$  is invertible in  $\Lambda$  whenever it is positive; note also that [Bach16] relies on the results of M. Levine also). Lastly, we explain that in all our results the categories  $DM_{\Lambda}(-)$  may be replaced by the categories  $D_{\Lambda}^{\text{MGl}}(-)$  of "cobordism-modules".

The author is deeply grateful to prof. Alexey Ananyevskiy and prof. Tom Bachmann for their really interesting comments.

## 1 Some preliminaries on motivic categories and related matters

In §1.1 we introduce some notation and a few conventions that we will use throughout the paper.

In §1.2 we discuss compactly generated triangulated categories along with their  $\Lambda$ -linear versions (for  $\Lambda \subset \mathbb{Q}$ , i.e., we invert some set  $S$  of primes in a triangulated category  $\underline{C}$  to obtain the corresponding  $\underline{C}_{\Lambda}$ ).

In §1.3 we recall some basics on the motivic categories  $SH(-)$  and  $DM(-)$ . We also note that these statements generalize to  $SH_{\Lambda}(-)$  and  $DM_{\Lambda}(-)$ . Moreover, we describe (abstract versions of) our basic continuity arguments.

In §1.4 we recall some well-known properties of the cohomological dimension of ("essentially finitely generated") fields and relate Grothendieck-Witt rings to  $SH(k)(S^0, S^0)$ .

### 1.1 Some notation and terminology

For categories  $C, D$  we write  $D \subset C$  if  $D$  is a full subcategory of  $C$ .

For a category  $C$  and  $X, Y \in \text{Obj } C$ , the set of  $C$ -morphisms from  $X$  to  $Y$  will be denoted by  $C(X, Y)$ .

Below  $\underline{C}$  will always denote a triangulated category.

For  $E \in \text{Obj } \underline{C}$  we will say that it is *2-torsion* if there exists  $t > 0$  such that  $2^t \text{id} E = 0$ .

We will use the term *exact functor* for a functor of triangulated categories (i.e., for a functor that preserves the structures of triangulated categories).

For a triangulated category  $\underline{C}$  and some  $D \subset \text{Obj } \underline{C}$  we will call the smallest subclass  $D'$  of  $\text{Obj } \underline{C}$  that contains  $D$  and is closed with respect to all  $\underline{C}$ -extensions and retractions the *envelope* of  $D$  (so, it is thick if  $D[1] \cong D$ ).

Below  $k$  and  $F$  will always be perfect fields of characteristic  $p$  (and the case  $p = 0$  will be the most interesting for us);  $k$  will usually be "the base field" for our motivic categories (whereas  $F$  will often run through all perfect fields).  $L$  will denote a field of characteristic  $p$  also; we will not assume  $L$  to be perfect (by default).

When writing  $k = \varinjlim k_i$  we will always assume that  $k_i$  are perfect fields also and the colimit is filtering; the corresponding morphisms  $k_i \rightarrow k$  will be denoted by  $m_i$ .

The category of perfect fields will be denoted by  $\mathcal{P}\mathcal{F}i$ .

For  $m : k \rightarrow k'$  being a  $\mathcal{P}\mathcal{F}i$ -morphism and  $E \in \text{Obj } C(k)$  the object  $C(m)(E)$  of  $C(k')$  will often be denoted by  $E_{k'}$ . If an object  $E'$  of  $C(k')$  is isomorphic to  $E_{k'}$  (for some  $E \in \text{Obj } C(k)$ ) then we will say that  $E'$  is *defined over  $k$* .

For a 2-functor  $C$  from  $\mathcal{P}\mathcal{F}i$  into a certain 2-category of categories (that will actually be the 2-category of tensor triangulated categories for all the examples of this paper) the *continuity property for morphisms* in  $C$  is the following statement:  $C(k)(M_k^0, N_k^0) \cong \varinjlim_i C(k_i)(M_{k_i}^0, N_{k_i}^0)$  whenever  $k = \varinjlim_i k_i$ , all these fields are extensions of a certain perfect  $k_0$ , whereas  $M^0$  and  $N^0$  are some objects of  $C(k_0)$ . This property is (an important) part of the following *continuity* property for  $C$  (cf. §4.3 of [CiD12]): we will say that  $C$  is continuous if we have  $C(k) \cong \varinjlim C(k_i)$  whenever  $k = \varinjlim k_i$  (i.e., we consider the 2-category colimit with the transition functors being the result of applying  $C$  to the corresponding  $\mathcal{P}\mathcal{F}i$ -morphisms).

We will say that  $k$  is non-orderable whenever  $-1$  is a sum of squares in it.

$SmVar$  will denote the set of (not necessarily connected) smooth  $k$ -varieties (and in some occasions we will consider  $SmVar$  as a category). More generally,  $SmVar(F)$  will denote the set of smooth  $F$ -varieties.

$\text{pt}$  will always denote the point  $\text{Spec } k$  (over  $k$ );  $\mathbb{P}^1$  will denote the projective line  $\mathbb{P}^1(k)$ , and  $\mathbb{A}^1 = \mathbb{A}^1(k)$  is the affine line.

## 1.2 On compactly generated categories and localizations of coefficients

In this subsection  $\underline{\mathcal{C}}$  will denote a triangulated category closed with respect to all small coproducts. We recall the following (more or less) well-known definitions.

**Definition 1.2.1.** 1. We will say that an object  $M$  of  $\underline{\mathcal{C}}$  is *compact* whenever the functor  $\underline{\mathcal{C}}(M, -)$  respects coproducts.

2. We will say that a set  $\{C_i\} \subset \text{Obj } \underline{\mathcal{C}}$  generates a subcategory  $\underline{\mathcal{D}} \subset \underline{\mathcal{C}}$  as a *localizing subcategory* if  $\underline{\mathcal{D}}$  equals the smallest full strict triangulated subcategory of  $\underline{\mathcal{C}}$  that is closed with respect to small coproducts and contains all  $C_i$ .

We will say that  $C_i$  generate  $\underline{\mathcal{C}}$  as its own localizing subcategory if we can take  $\underline{\mathcal{D}} = \underline{\mathcal{C}}$  in this definition.

3. We will say that  $C_i$  compactly generate  $\underline{\mathcal{C}}$  if all  $C_i$  are compact (in  $\underline{\mathcal{C}}$ ) and  $\{C_i\} \subset \text{Obj } \underline{\mathcal{C}}$  generate  $\underline{\mathcal{C}}$  as its own localizing subcategory.

We will say that  $\underline{\mathcal{C}}$  is compactly generated whenever there exists some set of compact generators of this sort.

*Remark 1.2.2.* Recall (see Lemma 4.4.5 of [Nee01]) that if  $C_i$  compactly generate  $\underline{\mathcal{C}}$  then the full subcategory  $\underline{\mathcal{C}}^c$  of compact objects of  $\underline{\mathcal{C}}$  is the smallest thick subcategory of  $\underline{\mathcal{C}}$  containing  $C_i$  (i.e., if  $\text{Obj } \underline{\mathcal{C}}^c$  is the envelope of  $\{C_i[j] : j \in \mathbb{Z}\}$  in the sense described in §1.1).

In the current paper we use the "homological convention" for  $t$ -structures (following [Mor03] and [Bach16]). Thus a  $t$ -structure  $t$  for  $\underline{\mathcal{C}}$  yields homological functors  $H_j^t$  from  $\underline{\mathcal{C}}$  to the heart  $\underline{Ht}$  of  $t$  such that  $H_j^t = H_0^t \circ [-j]$  for any  $j \in \mathbb{Z}$ . If  $t$  is non-degenerate (i.e., the collection  $\{H_j^t\}$  for  $j \in \mathbb{Z}$  is conservative; we will call these functors  $t$ -homology) then  $E \in \underline{\mathcal{C}}_{t \leq 0}$  (resp.  $E \in \underline{\mathcal{C}}_{t \geq 0}$ ) if and only if  $H_j^t(E) = 0$  for all  $j > 0$  (resp.  $j < 0$ ).

We recall the following existence statement.

**Proposition 1.2.3.** *Let  $\{C_i\} \subset \text{Obj } \underline{\mathcal{C}}$  be a set of compact objects. Then there exists a unique  $t$ -structure  $t$  for  $\underline{\mathcal{C}}$  such that  $\underline{\mathcal{C}}_{t \geq 0}$  is the smallest subclass of  $\text{Obj } \underline{\mathcal{C}}$  that contains  $\{C_i\}$  and is stable with respect to extensions, the suspension  $[+1]$ , and arbitrary (small) coproducts.*

*Proof.* This is Theorem A.1. of [TLS03].

□

*Remark 1.2.4.* 1. Recall that  $E \in \text{Obj } \underline{C}$  determines its  $t$ -decomposition triangle  $E_{t \geq 0} \rightarrow E \rightarrow E_{t \leq -1}$  (with  $E_{t \geq 0} \in \underline{C}_{t \geq 0}$  and  $E_{t \leq -1} \in \underline{C}_{t \leq -1} = \underline{C}_{t \leq 0}[-1]$ ) in a functorial way.

2. Under the assumptions of the proposition we will say that the  $t$ -structure  $t$  is generated by  $C_i$ .

3. If  $\{C_i\}$  is suspension-stable (i.e.,  $\{C_i\}[1] = \{C_i\}$ ) then the classes  $\underline{C}_{t \geq 0}$  and  $\underline{C}_{t \leq 0}$  are suspension-stable also. Thus  $\underline{C}_{t \geq 0}$  is the class of objects of the localizing subcategory  $\underline{D}$  generated by  $\{C_i\}$ , and  $E \mapsto E_{t \geq 0}$  yields the right adjoint to the embedding  $\underline{D} \rightarrow \underline{C}$ . This functor is certainly exact; this setting is called the *Bousfield localization* one (in [Nee01]).

We also recall some basics on "localizing coefficients" in a triangulated category.

Below  $S \subset \mathbb{Z}$  will always be a set of prime numbers;  $\mathbb{Z}[S^{-1}]$  will be denoted by  $\Lambda$ . We will often assume that  $S$  contains  $p$  whenever  $p > 0$ .

**Proposition 1.2.5.** *Assume that  $\underline{C}$  is compactly generated by its small subcategory  $\underline{C}'$ . Denote by  $\underline{C}_{S\text{-tors}}$  the localizing subcategory of  $\underline{C}$  (compactly) generated by  $c' \xrightarrow{\times s} c'$  for  $c' \in \text{Obj } \underline{C}'$ ,  $s \in S$ .*

*Then the following statements are valid.*

1.  $\underline{C}_{S\text{-tors}}$  also contains all cones of  $c \xrightarrow{\times s} c$  for  $c \in \text{Obj } \underline{C}$  and  $s \in S$ .
2. The Verdier quotient category  $\underline{C}_\Lambda = \underline{C}/\underline{C}_{S\text{-tors}}$  exists (i.e., the morphism groups of the localization are sets); the localization functor  $l: \underline{C} \rightarrow \underline{C}_\Lambda$  respects all coproducts and converts compact objects into compact ones. Moreover,  $\underline{C}_\Lambda$  is generated by  $l(\text{Obj } \underline{C}')$  as a localizing subcategory.
3. For any  $c \in \text{Obj } \underline{C}$ ,  $c' \in \text{Obj } \underline{C}'$ , we have  $\underline{C}_\Lambda(l(c'), l(c)) \cong \underline{C}(c, c') \otimes_{\mathbb{Z}} \Lambda$ .
4.  $l$  possesses a right adjoint  $G$  that is a full embedding functor. The essential image of  $G$  consists of those  $M \in \text{Obj } \underline{C}$  such that  $s \cdot \text{id}_M$  is an automorphism for any  $s \in S$  (i.e.,  $G(\underline{C})$  is essentially the maximal full  $\Lambda$ -linear subcategory of  $\underline{C}$ ).

*Proof.* See Proposition A.2.8 and Corollary A.2.13 of [Kel12] (cf. also Appendix B of [Lev13]).

□



*Remark 1.2.6.* For  $S = \{l\}$  (i.e., consisting of a single prime) we will write  $\underline{C}[l^{-1}]$  instead of  $\underline{C}_{\mathbb{Z}[l^{-1}]}$ .

Besides, for a triangulated category  $\underline{C}$  being a value of a 2-functor  $\underline{D}$  from  $\mathcal{P}\mathcal{F}i$  (i.e., if  $\underline{C} = \underline{D}(F)$  for some perfect field  $F$ ) its  $\Lambda$ -linear version will be denoted by  $\underline{D}_\Lambda(F)$  instead of  $\underline{D}(F)_\Lambda$ .

### 1.3 On motivic categories and continuity

Now we recall some basics properties of triangulated motivic categories (that were defined by Voevodsky and Morel). For our purposes it will be sufficient to consider them over perfect fields only; yet note that a much more general theory is currently available (thanks to the works of Ayoub, Cisinski, and Déglise). Respectively, instead of morphisms of base schemes we will consider morphisms of fields. Note also that (in contrast to the "usual" convention) the tensor product on (various versions of)  $SH(-)$  will be denoted by  $\otimes$ .

**Proposition 1.3.1.** *1. There exist covariant 2-functors  $k \mapsto SH(k)$  and  $k \mapsto DM(k)$  (the latter category was introduced in §4.2 of [Deg11]) from  $\mathcal{P}\mathcal{F}i$  into the 2-category of tensor triangulated categories.*

*The categories  $SH(k)$  and  $DM(k)$  are closed with respect to arbitrary small coproducts and the tensor products for them respect coproducts (when one of the arguments is fixed). Moreover, for a morphism  $m : k \rightarrow k'$  the functors  $SH(m)$  and  $DM(m)$  also respect all coproducts and the compactness of objects.*

*2. There exist exact functors  $SmVar \rightarrow SH(k) : X \mapsto \Sigma_T^\infty(X_+)$  (one may consider this as a notation) and  $M_{gm} : SmVar \rightarrow DM(k)$ ; they factor through the corresponding subcategories of compact objects  $SH(k)^c$  and  $DM^c(k)$ , respectively. Moreover, these two functors convert the products in  $SmVar$  into the tensor products in  $SH(k)$  and  $DM(k)$ , respectively, and convert the projection  $\mathbb{A}^1 \rightarrow \text{pt}$  into isomorphisms.*

*3. For any  $k$  there is an exact tensor functor  $M_k : SH(k) \rightarrow DM(k)$  (the motivization functor) that respects coproducts and the compactness of objects; we have  $M_k(\Sigma_T^\infty(X_+)) \cong M_{gm}(X)$  for any  $X \in SmVar$ .*

Besides, all the squares of the type

$$\begin{array}{ccc} SH(k) & \xrightarrow{M_k} & DM(k) \\ \downarrow SH(m) & & \downarrow DM(m) \\ SH(k') & \xrightarrow{M_{k'}} & DM(k') \end{array}$$

(for  $m : k \rightarrow k'$  being a  $\mathcal{P}\mathcal{F}i$ -morphism) are commutative.

4.  $M_k$  possesses a right adjoint  $U_k$  that respects coproducts. Furthermore,  $U_k \circ M_k$  is isomorphic to  $-\otimes H_{\mathbb{Z}}$  for a certain  $H_{\mathbb{Z}} \in \text{Obj } SH(k)$ .
5. The objects  $S^0 = \Sigma_T^\infty(\text{pt}_+)$  and  $\mathbb{Z} = M_{gm}(\text{pt})$  (we omit  $k$  in this notation) are tensor units of the corresponding motivic categories; we have  $DM(k)(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ .
6. Denote by  $T$  the complement to  $\Sigma_T^\infty(\text{pt}_+)$  in  $\Sigma_T^\infty(\mathbb{P}_+^1)$  (with respect to the natural splitting), and denote by  $\mathbb{Z}(1)[2]$  the complement to  $M_{gm}(\text{pt})$  in  $M_{gm}(\mathbb{P}^1)$ . Then these objects are  $\otimes$ -invertible in the corresponding categories, and  $M_k(T) \cong \mathbb{Z}(1)[2]$ .  
The  $i$ th iterates of the functors  $-\otimes (T[-1])$  and  $-\otimes (\mathbb{Z}(1)[1])$  will (abusively) be denoted by  $-\{i\}$  for all  $i \in \mathbb{Z}$ . Note here that the "usual" Tate twist  $-(1)$  (in the convention introduced by Voevodsky) certainly equals  $-\{1\} \circ [-1]$ .
7. The category  $SH(k)$  (resp.  $DM(k)$ ) is compactly generated (see Definition 1.2.1) by the objects  $\Sigma_T^\infty(X_+)\{i\}$  (resp.  $M_{gm}(X)\{i\}$ ) for  $X \in \text{SmVar}$ ,  $i \in \mathbb{Z}$ .
8. For any  $k$  there exists a canonical idempotent  $SH(k)$ -endomorphism  $\epsilon$  of  $S^0$  (see §6.1 of [Mor03]) such that  $M_k(\epsilon) = -id_{\mathbb{Z}}$ .
9. The 2-functors  $SH^c(-)$  and  $DM^c(-)$  are continuous in the sense described in §1.1 (i.e.,  $SH^c(k) \cong \varinjlim_i SH^c(k_i)$  and  $DM^c(k) \cong \varinjlim_i DM^c(k_i)$  whenever  $k = \varinjlim_i k_i$ ).

*Proof.* All of these assertions are rather well-known except possibly the first part of the last one, that can be found in Example 2.6(1) of [CiD15] (see also §6.1 and Remark 6.3.5 of [Mor03] for assertion 8). □

Since the concept of continuity is very important for the current paper, we will now make certain comments concerning it. Since we will need the  $\Lambda$ -linear versions of motivic triangulated categories (starting from §2.2 below), we will start from introducing them. As we will briefly explain, these categories are easily seen to fulfil the natural analogues of the properties listed in Proposition 1.3.1. In particular, one of the reasons for considering them is that they share all the continuity properties of  $SH^c(-)$  and  $DM^c(-)$ .

For the convenience of the readers we note that the following remarks are not necessary for the understanding of §2.1 (so, the reader may start from this section and return to them after that).

*Remark 1.3.2.* 1. For a set of primes  $S$  and  $\Lambda = \mathbb{Z}[S^{-1}]$  we will study the categories  $SH_\Lambda(k)$  and  $DM_\Lambda(k)$ ; see Proposition 1.2.5(2) (and the convention introduced in Remark 1.2.6). The properties of these categories are quite similar to that of  $SH(k)$  and  $DM(k)$  (as described in our Proposition 1.3.1) and easily follow from this proposition; our notation for  $\Lambda$ -linear motivic categories are obtained from the one introduced above by adding the lower index  $\Lambda$ .

Now we explain this in some more detail. Obviously, the correspondences  $F \mapsto SH_\Lambda(F)$  and  $M \mapsto DM_\Lambda(F)$  (for a perfect field  $F$ ) yield 2-functors from  $\mathcal{P}\mathcal{F}i$  into the 2-category of tensor triangulated categories, and the functors of the type  $SH_\Lambda(m)$  and  $DM_\Lambda(m)$  (for  $m$  being a morphism of perfect fields) respect the compactness of objects and all coproducts (see part 3 of our proposition). Besides, the tensor products for these categories respect coproducts (when one of the arguments is fixed).

Next, for the localization functor  $l : SH(k) \rightarrow SH_\Lambda(k)$  (resp.  $DM(k) \rightarrow DM_\Lambda(k)$ ) the object  $l(\Sigma_T^\infty(X_+))$  (resp.  $l(M_{gm}(X))$ ) will be denoted by  $\Sigma_{T,\Lambda}^\infty(X_+)$  (resp. by  $M_{gm,\Lambda}(X)$ ). We also note that the natural analogues of Proposition 1.3.1(5,6) are also valid. Hence  $\Sigma_{T,\Lambda}^\infty(X_+) \cong \Sigma_{T,\Lambda}^\infty((X \times \mathbb{A}^j)_+)$  and  $M_{gm,\Lambda}(X) \cong M_{gm,\Lambda}(X \times \mathbb{A}^j)$  for any  $j \geq 0$ . We also obtain that the objects  $\Sigma_{T,\Lambda}^\infty(X_+)\{i\}$  (resp.  $M_{gm,\Lambda}(X)\{i\}$ ) for  $i \in \mathbb{Z}$  and  $X \in SmVar$  compactly generate  $SH_\Lambda(k)$  (resp.  $DM_\Lambda(k)$ ), whereas the smallest thick subcategory of  $SH_\Lambda(k)$  (resp.  $DM_\Lambda(k)$ ) containing all these objects is exactly  $SH_\Lambda^c(k)$  (resp.  $DM_\Lambda^c(k)$ ).

Furthermore,  $M_k$  naturally induces an exact tensor functor  $M_{k,\Lambda}$  that also respects the compactness of objects and all coproducts. The restriction of  $M_{k,\Lambda}$  to the subcategory  $SH_\Lambda^c(k)$  of compact objects (with its image being the corresponding  $DM_\Lambda^c(k)$ ) will be denoted by  $M_{k,\Lambda}^c$ .

Combining these facts with Proposition 1.3.1(9) (along with Proposition 1.2.5(3)) we easily obtain that the 2-functors  $SH_\Lambda^c(-)$  and  $DM_\Lambda^c(-)$  are continuous in the sense described in §1.1.

2. We will need a certain property of continuity for families of subsets of  $\text{Obj } DM_\Lambda^c(-)$ . To avoid (minor) set-theoretical difficulties, till the end of the section will assume that  $DM_\Lambda^c(F)$  is a small category for any perfect field  $F$ . This technical assumption is easily seen not to affect the results below (and we may actually adopt it in the rest of this paper also).

So, let  $O$  be a subfunctor of the functor  $\text{Obj } DM_\Lambda^c$  from  $\mathcal{P}\mathcal{F}i$  to the category of sets (i.e.,  $O(F) \subset \text{Obj } DM_\Lambda^c(F)$  for all perfect  $F$ , and  $O(m)$  for a morphism  $m : k \rightarrow k'$  of perfect fields is given by the restriction of  $DM_\Lambda(m)$  onto  $O(k)$ ). Then we will say that  $O$  is  $DM_\Lambda^c$ -continuous if it satisfies the following condition: for  $k = \varinjlim k_i$  and any  $M \in O(k)$  there exists some  $k_0 \in \{k_i\}$  and  $M^0 \in O(k_0)$  such that  $M \cong M_k^0$  (i.e.,  $M \cong O(m_0)(M^0)$  for the corresponding  $m_0 : k_0 \rightarrow k$ ; see §1.1). Note that latter condition is certainly equivalent to the set of  $DM_\Lambda^c(k)$ -isomorphism classes in  $O(k)$  being the direct limit of the sets of  $DM_\Lambda^c(k_i)$ -isomorphism classes in  $O(k_i)$ .

3. Below we will apply the following consequence of continuity: for any  $DM_\Lambda^c$ -continuous  $O$  the property  $M_{k,\Lambda}^c(E) \in \text{Obj } SH_\Lambda^c(k)$  to belong to  $O(k)$  is "continuous" also. This means the following: if  $k = \varinjlim k_i$ ,  $E \in \text{Obj } SH_\Lambda^c(k)$ , and  $M_{k,\Lambda}(E) \in O(k)$ , then there exists some  $k_j \in \{k_i\}$  along with  $E^j \in SH_\Lambda^c(k_j)$  such that  $M_{k_j,\Lambda}(E^j) \in O(k_j)$  and  $E_k^j \cong E$  (see §1.1).

Indeed, the continuity property for  $SH_\Lambda^c(-)$  allows us to choose some  $k_0 \in \{k_i\}$  such that  $E$  is defined over it (i.e., such that there exists  $E^0 \in \text{Obj } SH_\Lambda^c(k_0)$  with  $E \cong E_k^0$ ). Next, the  $DM_\Lambda^c$ -continuity of  $O$  yields the existence of  $k_1 \in \{k_i\}$  and  $M^1 \in O(k_1)$  such that  $M_k^1 \cong M_{k,\Lambda}(E)$ . Furthermore, the continuity property for morphisms in  $DM_\Lambda^c(-)$  (see §1.1) yields the existence of  $k_2 \in \{k_i\}$  that contains both  $k_0$  and  $k_1$  such that  $(M_{k_0,\Lambda}(E^0))_{k_2} \cong M_{k_2}^1$ . Thus we can take  $k_j = k_2$ ,  $E^j = E_{k_2}^0$  (since  $M_{k_2,\Lambda}(E_{k_2}^0) \cong M_{k_2}^1 \in (O(k_1))_{k_2} \subset O(k_2)$ ).

4. Now we describe some "tools" for constructing  $DM_\Lambda^c$ -continuous functors.

Firstly, the functors  $F \mapsto \{0\} \subset \text{Obj } DM_\Lambda^c(F)$  and  $F \mapsto \{M_{gm,\Lambda}(SmVar(F))\{r\}[j]\}$  (for  $F$  being a perfect field and any fixed  $r, j \in \mathbb{Z}$ ) are obviously  $DM_\Lambda^c$ -continuous.

Next, the "union" of any set of continuous functors is easily seen to be continuous.

Lastly, if  $O$  is  $DM_\Lambda^c$ -continuous then the functor sending  $F$  into the envelope of  $O(F)$  (in  $DM_\Lambda^c(F)$ ) is  $DM_\Lambda^c$ -continuous also (recall that we assume  $DM_\Lambda^c(F)$  to be small).

## 1.4 On the cohomological dimension and Grothendieck-Witt rings of fields

As we have said in §1.1,  $L$  always denotes some (not necessarily perfect) characteristic  $p$  field. We recall the following well-known facts.

**Proposition 1.4.1.** *Let  $L$  be a finitely generated field (i.e.,  $L$  is finitely generated over its prime subfield). Then the following statements are valid.*

1. *If  $L$  is non-orderable then its cohomological dimension (at any prime) is finite.*
2. *The cohomological dimension of  $L$  (whether it is finite or not) equals the one of the perfect closure  $L^{perf}$  of  $L$ .*

*Proof.* 1. See [Ser13], §II.3.3 and II.4.2.

2. It suffices to note that the absolute Galois group of  $L$  equals the one of its perfect closure.

□

The following easy lemma follows immediately.

**Corollary 1.4.2.** *If  $k$  is non-orderable then it may be presented as a filtered direct limit of perfect fields of finite cohomological dimension.*

*Proof.* It suffices to present  $k$  (recall that we assume it to be perfect) as the direct limit of the perfect closures of its finitely generated subfields, and apply the previous proposition.

□

*Remark 1.4.3.* 1. Note however that below (everywhere except in §3.2) is will actually be sufficient to present  $k$  as the direct limit of fields of finite cohomological 2-dimension.

2. Recall that the virtual cohomological 2-dimension of a field  $L$  of characteristic  $\neq 2$  may be defined as the cohomological 2-dimension of  $L[\sqrt{-1}]$ . Thus any finitely generated field of characteristic  $\neq 2$  is of finite virtual cohomological 2-dimension.

Now we recall some basics on Grothendieck-Witt rings and their relation to  $SH(-)$ .

*Remark 1.4.4.* 1. As shown in §6.3 of [Mor03],  $SH(k)(S^0, S^0) \cong GW(k)$  (the Grothendieck-Witt of  $k$ ). If  $p \neq 2$  then the latter is the Grothendieck group of non-degenerate  $k$ -quadratic forms. It is isomorphic to the kernel of  $W(k) \oplus \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ , where  $W(k)$  is the Witt ring of (quadratic forms over  $k$ ) and the projection  $W(k) \rightarrow \mathbb{Z}/2\mathbb{Z}$  is given by the parity of the dimension of quadratic forms. In the case  $p = 2$  one should consider symmetric bilinear forms instead of quadratic ones here.

As mentioned in the beginning of §2 of [ArE01], if  $p \neq 2$  then the group  $W(k)$  is an extension of the free group whose generators correspond to orderings on  $k$  by a torsion group. Thus the kernel of  $M_{k*} : SH(k)(S^0, S^0) \rightarrow DM(k)(\mathbb{Z}, \mathbb{Z})$  is torsion if and only if  $k$  is non-orderable (at least, in the case  $p \neq 2$ ).

2. It is no wonder that structural results on Witt rings of fields play a very important role in motivic homotopy theory. In particular, they were crucial for [Bach16], [Lev13], and [Aso16]. Information of this sort was also actively used in the previous version of the current paper; yet the corresponding arguments were essentially incorporated in the new version of [Bach16] (when Lemma 18 was added to it).

## 2 The conservativity of motivization results

The main result of this section is that (the "compact version" (b) of) Theorem 15 of [Bach16] can be extended to the case when  $k$  is an arbitrary non-orderable field. Moreover, the restriction of  $M_k^c$  to 2-torsion objects is conservative for any  $k$ .

So, in §2.1 we prove the "triangulated parts" of these results. We deduce them from similar results of *ibid.* (where certain cohomological dimension finiteness was assumed) using a simple continuity argument (that is a particular case of the reasoning described in Remark 1.3.2(3)). We also note that the conservativity of  $M_k^c$  never extends to "the whole"  $M_k$ ; also,  $M_k^c$  is never conservative if  $k$  is not non-orderable (i.e., if it is formally real). Moreover, Morel's morphism  $\eta$  is torsion if and only if  $k$  is non-orderable; this is also equivalent to the vanishing of the component  $SH^-(k)$  of the Morel's decomposition (here one can consider either its  $\mathbb{Z}[\frac{1}{2}]$ -linear or the  $\mathbb{Q}$ -linear version).

In §2.2 we study the homotopy  $t$ -structures and (Voevodsky's) slice filtrations for  $SH_\Lambda(-)$  and  $DM_\Lambda(-)$  (for the coefficient ring  $\Lambda \subset \mathbb{Q}$ ); their properties follow from their well-known  $\mathbb{Z}$ -linear versions.

In §2.3 we prove the  $\Lambda$ -linear version of (the stronger part of) Bachmann's theorem over an arbitrary non-orderable  $k$ , stating that the  $m$ -homotopy connectivity of  $M_{k,\Lambda}(E)$  for  $E \in \text{Obj } SH_\Lambda^c(k)$  ensures the  $m$ -homotopy connectivity (with respect to the homotopy  $t$ -structure  $t_\Lambda^{SH}$ ) of  $E$  itself. We also give the following immediate applications of this result (for  $k$  being any non-orderable perfect field): we prove the corresponding generalization of Theorem 2.2.1 of [Aso16], and prove that Theorem 30 of [Bach15] (on the  $\otimes$ -invertibility of certain motives of affine quadrics) may be carried over to motivic spectra.

## 2.1 On "triangulated conservativity" in the $\mathbb{Z}$ -linear setting

**Proposition 2.1.1.** *I Let  $k$  be a non-orderable field. Then the following statements are valid.*

1. *There exists  $N \geq 0$  such that  $2^N(1 + \epsilon) = 0$  in  $SH(k)$  (see Proposition 1.3.1(8)) and  $2^N\eta = 0$ , where  $\eta$  is the Morel's morphism  $S^0\{1\} \rightarrow S^0$ .*
  2. *The restriction  $M_k^c$  of the functor  $M_k$  to  $SH(k)^c$  is conservative.*
- II Let  $E$  be a 2-torsion object (see §1.1) of  $SH(k)^c$ . Then  $E = 0$  whenever  $M_k(E) = 0$ .*

*Proof.* I.1. By Lemma 6.7 of [Lev13], the assertion is fulfilled if  $p > 0$ . Thus we can assume  $p \neq 2$ .

Now,  $1 + \epsilon$  belongs to the image in  $SH(k)(S^0, S^0) \cong GW(k)$  of the class  $[x^2] - [-x^2]$ ; see Remark 1.4.4(1). Hence the first part of the assertion easily follows from Proposition 1.4.1. The second part of the assertion follows from the first one immediately by Lemma 6.2.3 of [Mor03].

2. According to Theorem 15 of [Bach16] (see version (b) of the first assertion in the theorem), the statement is valid if the cohomological 2-dimension of  $k$  is finite.

Next, in the general case Corollary 1.4.2 enables us to present  $k$  as  $\varinjlim k_i$  (recall here the conventions described in §1.1) so that the cohomological 2-dimensions of  $k_i$  are finite. Thus to finish the proof suffices to recall that the

correspondence  $F \mapsto \{0\} \subset \text{Obj } DM^c(F)$  is  $DM^c$ -continuous (see Remark 1.3.2(4)); here we take  $\Lambda = \mathbb{Z}$ ) and apply part 3 of this remark.

Now we explain this continuity argument in our concrete situation (for the sake of those readers that have problems with Remark 1.3.2).

Assume that  $M_k(E) = 0$  for some  $E \in \text{Obj } DM^c(k)$ . By the continuity property for  $SH^c(-)$  (see Proposition 1.3.1(9)) there exists  $k_0 \in \{k_i\}$  such that  $E$  is defined over  $k_0$  (i.e., there exists  $E^0 \in \text{Obj } SH^c(k_0)$  such that  $E_k^0 \cong E$ ; cf. §1.1). Next, the continuity property for the morphisms in  $DM^c(-)$  (see §1.1) yields the existence of  $k_1 \in \{k_i\}$  such that  $k_1$  is an extension of  $k_0$  and  $M_{k_1}(E_{k_1}^0) = 0$ . Hence applying Theorem 15 of [Bach16] to  $E_{k_1}^0$  we obtain  $E_{k_1}^0 = 0$ . Thus  $E \cong (E_{k_1}^0)_k$  is zero also.

II The proof is rather similar to that of assertion I.2.

Firstly, this assertion enables us to assume that  $k$  is formally real; so, we restrict ourselves to the case  $p = 0$ .

Then  $k = \lim k_i$  for  $k_i$  being (perfect) finitely generated fields. Similarly to the previous proof, the continuity property for  $SH^c(-)$  yields the existence of  $k_0 \in \{k_i\}$  and  $E^0 \in \text{Obj } SH^c(k_0)$  such that  $E_k^0 \cong E$ . Moreover, the continuity property for morphisms in  $SH^c(-)$  enables us to assume that  $E_0$  is 2-torsion.

Thus it suffices to prove our assertion for  $k$  being an (orderable) perfect finitely generated field. Hence it remains to apply Lemma 18 of *ibid* (along with Remark 1.4.3(2)).

□

*Remark 2.1.2.* 1. Combining part I.1 of our proposition with Lemma 6.8 of [Lev13] we immediately obtain that  $SH^+(k)$  (as defined in *loc. cit.*) is isomorphic to  $SH(k)[\frac{1}{2}]$  whenever  $k$  is a non-orderable field. On the other hand, if  $k$  is formally real (i.e., not non-orderable) then this property fails (and moreover, the complementary category  $SH^-(k)$  is "non-torsion"). The latter can be easily seen from the isomorphism  $SH^+(k)_{\mathbb{Q}} \cong DM(k)_{\mathbb{Q}}$  (see Theorem 16.2.13 of [CiD12]) whereas  $SH(k)(S^0, S^0) \otimes \mathbb{Q} \not\cong DM(k)(\mathbb{Z}, \mathbb{Z}) \otimes \mathbb{Q}$  in this case as we have just noted.

Since  $SH(k)[\frac{1}{2}]$  may be considered as a subcategory of  $SH(k)$  (see Proposition 1.2.5(4)), this observation demonstrates that the extension of scalars functors  $SH(-)$  are far from being conservative in general; thus various continuity-type properties of motivic spectra (as described in the current paper) are "the best one can hope for".

2. The "whole"  $M_k$  is not conservative for any (perfect)  $k$ . Indeed,



consider the homotopy colimit  $S^0[\eta^{-1}]$  of the sequence of morphisms  $S^0 \xrightarrow{\eta\{-1\}} S^0\{-1\} \xrightarrow{\eta\{-2\}} S^0\{-2\} \rightarrow \dots$  (originally considered in Definition 2 of [ANL15]; note yet that the definition of  $\eta$  in *ibid.* differs from our one by  $-\{1\}$ ). Since  $M_k(\eta) = 0$ , we have  $M_k(S^0[\eta^{-1}]) = 0$  (see Lemma 1.6.7 of [Nee01]). On the other hand, Theorem 1 of [ANL15] easily yields that  $S^0[\eta^{-1}] \neq 0$ .

Thus the kernel of "the whole"  $M_k$  is not generated by the one of  $M_k^c$  (as a localizing subcategory of  $SH(k)$ ) in general.

3. If  $-1$  is not a sum of squares in  $k$  (and so  $k$  is formally real) then the kernel of  $M_k^c$  is non-zero. Indeed, the object  $C = \text{Cone}(2id_{S^0(k)} + \epsilon)$  is certainly compact, and the long exact sequence  $\dots SH(S^0, S^0) \cong GW(k) \xrightarrow{\times(2[x^2] - [-x^2])} SH(S^0, C) \rightarrow SH(S^0, S^0[1]) = \{0\}$  (see Remark 1.4.4(1)) easily yields that  $C \neq 0$  (since considering the split surjection of  $GW(k)$  onto  $\mathbb{Z}$  corresponding to any ordering on  $k$  one obtains  $SH(S^0, C) \supset \mathbb{Z}/3\mathbb{Z}$ ). Yet  $M_k(C) = 0$  since  $M_k(2id_{S^0(k)} + \epsilon) = id_{\mathbb{Z}}$ .

4. Certainly, for any  $E \in \text{Obj } SH_{\Lambda}^c(k)$  a cone  $E[2]$  of  $E \xrightarrow{2id_E} E$  is a 2-torsion object (that is surely annihilated by 4). Thus part II of the proposition above yields that  $M_k(E)$  can vanish only if (the endomorphism ring of)  $E$  is uniquely 2-divisible. So it seems reasonable to conjecture that  $M_k^c(E)$  vanishes (assuming that  $2 \notin S$ ) only for  $E$  being an odd torsion object.

On the other hand, the odd torsion in the kernel of  $M_k^c$  may be quite "large" if  $k$  is formally real. In particular,  $M_k$  kills  $C \otimes \text{Obj } SH(k)$ , where  $C$  is the object constructed above. Note that one can also easily construct  $l$ -torsion objects "similar to  $C$ " for  $l$  being any odd integer.

Furthermore, note that the elements of the kernel of  $M_{k,\Lambda}^c$  are uniquely 2-divisible (i.e., are  $\mathbb{Z}[2^{-1}]$ -linear) for any choice of  $S$  (and so, of  $\Lambda$ ) by Corollary 2.3.2 below. We conjecture that this kernel consists of odd torsion elements only whenever  $2 \notin S$ .

5. Proposition 2.1.1(I.1) is certainly not quite new; cf. Remark 1.2.8(2) of [Deg13].

## 2.2 More auxiliary results: homotopy $t$ -structures, slice filtrations, and their continuity

As always,  $S$  will denote some set of primes,  $\Lambda = \mathbb{Z}[S^{-1}]$ . Starting from this section we will freely use the notation and results described in Remark 1.3.2.

**Definition 2.2.1.** 1. Denote by  $t_{\Lambda}^{SH}$  (resp.  $t_{\Lambda}^{DM}$ ) the  $t$ -structure on  $SH_{\Lambda}(k)$  (resp. on  $DM_{\Lambda}(k)$ ) generated by  $\Sigma_{T,\Lambda}^{\infty}(X_+)\{i\}$  (resp. by  $M_{gm,\Lambda}(X)\{i\}$ ) for  $X \in SmVar$ ,  $i \in \mathbb{Z}$  (see Remark 1.2.4(2)). We will call these  $t$ -structures *homotopy* ones.

We will say that  $E \in \text{Obj } SH_{\Lambda}(k)$  is *homotopy connective* if it belongs to  $SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq i}$  for some  $i \in \mathbb{Z}$ .

2. Denote by  $SH_{\Lambda}^{eff}(k)$  (resp.  $DM_{\Lambda}^{eff}(k)$ ) the localizing subcategory of  $SH_{\Lambda}(k)$  (resp. of  $DM_{\Lambda}(k)$ ) generated by  $\Sigma_{T,\Lambda}^{\infty}(X_+)$  (resp. by  $M_{gm,\Lambda}(X)$ ; so, we follow the convention introduced in Remark 1.2.6).

Obviously,  $SH_{\Lambda}^{eff}(k)\{1\} = SH_{\Lambda}^{eff}(k)(1) \subset SH_{\Lambda}^{eff}(k)$  and  $DM_{\Lambda}^{eff}(k)\{1\} = DM_{\Lambda}^{eff}(k)(1) \subset DM_{\Lambda}^{eff}(k)$ ; we will call the filtration of  $SH_{\Lambda}(k)$  by  $SH_{\Lambda}^{eff}(k)\{i\}$  (resp. of  $DM_{\Lambda}(k)$  by  $DM_{\Lambda}^{eff}(k)\{i\}$ ) for  $i \in \mathbb{Z}$  the *slice* filtration. We will say that the elements of  $\cap_{i \in \mathbb{Z}} \text{Obj } SH_{\Lambda}^{eff}(k)\{i\}$  and of  $\cap_{i \in \mathbb{Z}} \text{Obj } DM_{\Lambda}^{eff}(k)\{i\}$  are *infinitely effective*.

We will say that  $E \in \text{Obj } SH_{\Lambda}(k)$  is *slice-connective* if it belongs to  $\text{Obj } SH_{\Lambda}^{eff}(i)$  for some  $i \in \mathbb{Z}$ .

We will omit  $\Lambda$  in this notation if  $\Lambda = \mathbb{Z}$ .

*Remark 2.2.2.* 1. For any  $X \in SmVar$  we have  $\Sigma_{T,\Lambda}^{\infty}(X_+) \in \text{Obj } SH_{\Lambda}^{eff}(k) \cap SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq 0}$  and  $M_{gm,\Lambda}(X) \in \text{Obj } DM_{\Lambda}^{eff}(k) \cap DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq 0}$ . Hence for any compact object  $E$  of  $SH_{\Lambda}(k)$  (resp. of  $DM_{\Lambda}(k)$ ) there exists  $r \in \mathbb{Z}$  such that  $E$  belongs to  $\text{Obj } SH_{\Lambda}^{eff}(k)\{r\} \cap SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq r}$  (resp. to  $\text{Obj } DM_{\Lambda}^{eff}(k)\{r\} \cap DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq r}$ ); here we apply Remark 1.2.2.

2. In [Bach16] the objects that we call homotopy connective were called just connective.

Now let us establish some more basic properties of these filtrations (and recall that  $SH(k)^c[\frac{1}{p}]$  is rigid).

**Proposition 2.2.3.** *Let  $r \in \mathbb{Z}$ ,  $m : k \rightarrow k'$  is an embedding of perfect fields. Then the following statements are valid.*

1.  $SH_{\Lambda}(m)$  sends  $SH_{\Lambda}^{eff}(k)\{r\}$  into  $SH_{\Lambda}^{eff}(k')\{r\}$  and maps  $SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq r}$  into  $SH_{\Lambda}(k')_{t_{\Lambda}^{SH} \geq r}$ .
2.  $DM_{\Lambda}(m)$  sends  $DM_{\Lambda}^{eff}(k)\{r\}$  into  $DM_{\Lambda}^{eff}(k')\{r\}$  and maps  $DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq r}$  into  $DM_{\Lambda}(k')_{t_{\Lambda}^{DM} \geq r}$ .
3.  $M_{k,\Lambda}$  sends  $SH_{\Lambda}^{eff}(k)\{r\}$  into  $DM_{\Lambda}^{eff}(k)\{r\}$  and maps  $SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq r}$  into  $DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq r}$ .

4.  $\text{Obj } SH_{\Lambda}^{eff}(k)\{r\} \otimes \text{Obj } SH_{\Lambda}^{eff}(k) \subset \text{Obj } SH_{\Lambda}^{eff}(k)\{r\}$  and  $SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq r} \otimes SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq 0} \subset SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq r}$ ;  $\text{Obj } DM_{\Lambda}^{eff}(k)\{r\} \otimes \text{Obj } DM_{\Lambda}^{eff}(k) \subset \text{Obj } DM_{\Lambda}^{eff}(k)\{r\}$  and  $DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq r} \otimes DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq 0} \subset DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq r}$ .
5. The correspondences  $F \mapsto \text{Obj } DM_{\Lambda}^{eff}(-)\{r\} \cap \text{Obj } DM_{\Lambda}^c(-)$  and  $F \mapsto DM_{\Lambda}(-)_{t_{\Lambda}^{DM} \geq r} \cap \text{Obj } DM_{\Lambda}^c(-)$  for  $F \in \text{Obj } \mathcal{P}\mathcal{F}i$  are  $DM_{\Lambda}^c$ -continuous in the sense of Remark 1.3.2(2), i.e., if  $k = \varinjlim k_i$  and  $E \in \text{Obj } DM_{\Lambda}^{eff}(k)\{r\} \cap \text{Obj } DM_{\Lambda}^c(k)$  (resp.  $E \in DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq r} \cap \text{Obj } DM_{\Lambda}^c(k)$ ) then there exists  $k_0 \in \{k_i\}$  along with some  $E^0 \in \text{Obj } DM_{\Lambda}^{eff}(k_0)\{r\} \cap \text{Obj } DM_{\Lambda}^c(k_0)$  (resp.  $E^0 \in DM_{\Lambda}(k_0)_{t_{\Lambda}^{DM} \geq r} \cap \text{Obj } DM_{\Lambda}^c(k_0)$ ) such that  $E_k^0 \cong E$ .
6. The  $t$ -structures  $t_{\Lambda}^{SH}$  and  $t_{\Lambda}^{DM}$  are non-degenerate.
7. Denote by  $H_{\Lambda}$  the image of  $H_{\mathbb{Z}}$  (see Proposition 1.3.1(3)) in  $SH_{\Lambda}$ . Then all morphisms from  $SH_{\Lambda}^{eff}(k)\{1\}$  into  $H_{\Lambda}$  are zero ones, and there exists a (natural) morphism  $S_{\Lambda}^0 = \Sigma_{T,\Lambda}^{\infty}(\text{pt}_+) \rightarrow H_{\Lambda}$  whose cone  $\overline{H}_{\Lambda}$  belongs to  $SH_{\Lambda}^{eff}(k)\{1\}$ .
8. In the case  $p > 0$  assume in addition that  $p \in S$ . Then any infinitely effective object of  $DM_{\Lambda}^c(k)$  (see Definition 2.2.1(2)) is zero.
9. Assume once again that  $S$  contains  $p$  if  $p > 0$ . Then the categories  $SH_{\Lambda}^c(k)$  and  $DM_{\Lambda}^c(k)$  are rigid (i.e., all their objects are dualizable). Moreover,  $SH_{\Lambda}^c(k)$  is the smallest thick subcategory of  $SH_{\Lambda}(k)$  containing all  $\Sigma_{T,\Lambda}^{\infty}(P_+)\{i\}$  for  $P$  being smooth projective over  $k$  and  $i \in \mathbb{Z}$ ;  $DM_{\Lambda}^c(k)$  is the smallest thick subcategory of  $DM_{\Lambda}(k)$  containing all  $M_{gm,\Lambda}(P)\{i\}$ .
10. All morphisms from  $S_{\Lambda}^0$  into  $SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq 1}$  are zero ones.

*Proof.* 1, 2, 3. By the definition of the corresponding classes, it suffices to note that  $SH_{\Lambda}(m)$ ,  $DM_{\Lambda}(m)$ , and  $M_{k,\Lambda}$  are exact functors that respect small coproducts.

4. Since the tensor product bi-functors for  $SH_{\Lambda}(k)$  and  $DM_{\Lambda}(k)$  respect co-products when one of the arguments is fixed and also "commute with  $-\{i\}$ ", it suffices to note that  $\Sigma_{T,\Lambda}^{\infty}(-+)(SmVar) \otimes \Sigma_{T,\Lambda}^{\infty}(-+)(SmVar) \subset \Sigma_{T,\Lambda}^{\infty}(-+)(SmVar)$  and  $M_{gm,\Lambda}(SmVar) \otimes M_{gm,\Lambda}(SmVar) \subset M_{gm,\Lambda}(SmVar)$ .

5. We can certainly assume  $r = 0$ . Next, for any perfect field  $F$  Remark 1.2.2 yields that  $DM_{\Lambda}^{eff}(F) \cap \text{Obj } DM_{\Lambda}^c(F)$  is the smallest thick subcategory

of  $DM_\Lambda(F)$  containing  $M_{gm,\Lambda}(X)$  for all  $X \in SmVar(F)$ . Moreover, Theorem 3.7 of [PoS16] (as well as the more general Theorem 4.2.1 of [Bon16]) implies that  $DM_\Lambda(F)_{t_\Lambda^{DM} \geq 0} \cap \text{Obj } DM_\Lambda^c(F)$  is the  $SH_\Lambda(F)$ -envelope (see §1.1) of  $M_{gm,\Lambda}(X)\{j\}[l]$  for  $X$  running through  $SmVar(F)$ ,  $j \in \mathbb{Z}$ , and  $l \geq 0$ . Hence the assertion follows from Remark 1.3.2(4).

6. The statement easily reduces to the case  $\Lambda = \mathbb{Z}$  in which it is well-known.

7. Obviously it suffices to prove the statement for  $\Lambda = \mathbb{Z}$ . In this case it is given by Theorem 10.5.1 of [Lev08] (if one combines it with the trivial Remark 2.2.5 below).

8. Immediate from Proposition 2.2.6 of [BoS14] (as well as from Proposition 2.2.6(3) below).

9. Immediate from Theorem 2.4.8 of [BoD15] (that relies on Appendix B of [LYZR16]); cf. also Lemma 2.3.1 of [Bon11] and Proposition 5.5.3 of [Kel12] where independent proofs of the  $DM_\Lambda^c(k)$ -part of the assertion were given.

10. This is a well-known statement that easily follows from Example 5.2.2 of [Mor03]. □

We will also need the effective versions of our homotopy  $t$ -structures along with some of their properties.

**Definition 2.2.4.** 1. Denote by  $t_\Lambda^{SH^{eff}}$  (resp.  $t_\Lambda^{DM^{eff}}$ ) the  $t$ -structure on  $SH_\Lambda^{eff}(k)$  (resp. on  $DM_\Lambda^{eff}(k)$ ) generated by  $\Sigma_{T,\Lambda}^\infty(X_+)$  (resp. by  $M_{gm,\Lambda}(X)$ ) for  $X \in SmVar$ .

2. Denote by  $i_\Lambda^{SH} = i_{\Lambda,k}^{SH}$  (resp. by  $i_\Lambda^{DM} = i_{\Lambda,k}^{DM}$ ) the embedding  $SH_\Lambda^{eff}(k) \rightarrow SH_\Lambda(k)$ . Their right adjoints (see Remark 1.2.4(3)) will be denoted by  $w_\Lambda^{SH}$  and  $w_\Lambda^{DM}$ , respectively.

Omitting  $k$ , let us denote the compositions  $w_\Lambda^{SH} \circ i_\Lambda^{SH}$  and  $w_\Lambda^{DM} \circ i_\Lambda^{DM}$  by  $\nu_{SH_\Lambda}^{\geq 0}$  and  $\nu_{DM_\Lambda}^{\geq 0}$ , respectively. Besides, for any  $r \in \mathbb{Z}$  we will consider the functors  $\nu_{SH_\Lambda}^{\geq r} = (\nu_{SH_\Lambda}^{\geq 0}(-\{-r\}))\{r\}$  and  $\nu_{DM_\Lambda}^{\geq r} = (\nu_{DM_\Lambda}^{\geq 0}(-\{-r\}))\{r\}$ .

3. For a homological functor  $H$  from  $SH_\Lambda(k)$  (resp. from  $DM_\Lambda(k)$ ) with values in some abelian category the symbol  $\text{Fil}_{\text{Tate}}^r H$  will (similarly to [Lev13]) denote the functor  $E \mapsto \text{Im}(H(\nu_{SH_\Lambda}^{\geq r}(E)) \rightarrow H(E))$  (resp.  $E \mapsto \text{Im}(H(\nu_{DM_\Lambda}^{\geq r}(E)) \rightarrow H(E))$ ; here the connecting morphisms are induced by the corresponding counits; see Remark 2.2.5).

*Remark 2.2.5.* Certainly, the functor  $\nu_{SH_\Lambda}^{\geq r}$  (resp.  $\nu_{DM_\Lambda}^{\geq r}$ ) gives a projection of  $SH_\Lambda(k)$  onto  $SH_\Lambda(k)\{r\}$  (resp. of  $DM_\Lambda(k)$  onto  $DM_\Lambda(k)\{r\}$ ). Moreover, the counits of the corresponding adjunctions yield natural transformations from  $\nu_{SH_\Lambda}^{\geq r}$  and  $\nu_{DM_\Lambda}^{\geq r}$  into the identity functors for  $SH_\Lambda(k)$  and  $DM_\Lambda(k)$ , respectively. Furthermore, for any  $E \in \text{Obj } SH_\Lambda(k)$  the counit morphism  $a_r(E) : \nu_{SH_\Lambda}^{\geq r}(E) \rightarrow E$  is certainly characterized by the following condition:  $\nu_{SH_\Lambda}^{\geq r}(E) \in \text{Obj } SH_\Lambda(k)\{r\}$  and there are only zero morphisms from  $\text{Obj } SH_\Lambda(k)\{r\}$  into  $\text{Cone}(a_r(E))$ .

Below we will apply the following obvious abstract nonsense observation: for any  $E \in \text{Obj } SH_\Lambda(k)$  (resp.  $M \in \text{Obj } DM_\Lambda(k)$ ) any morphism from  $SH_\Lambda^{eff}(k)\{r\}$  into  $E$  (resp. from  $DM_\Lambda^{eff}(k)\{r\}$  into  $M$ ) factors through  $a_r(E)$  (resp. through the corresponding counit morphism for  $M$ ).

The following statements appear to be (quite easy and) rather well-known.

**Proposition 2.2.6.** 1. *The functors  $i_\Lambda^{SH}$  and  $i_\Lambda^{DM}$  are right  $t$ -exact with respect to the corresponding  $t$ -structures, whereas their right adjoints are  $t$ -exact.*

2. *The spectrum  $\overline{H}_\Lambda$  (see Proposition 2.2.3(7)) belongs to  $SH_\Lambda^{eff}(k)_{t_\Lambda^{SHeff} \geq 1}$ .*

3. *The  $\Lambda$ -linear analogue  $U_{k,\Lambda}$  of  $U_k$  sends  $DM_\Lambda^{eff}(k)\{r\}$  into  $SH_\Lambda^{eff}(k)\{r\}$  and maps  $DM_\Lambda(k)_{t_\Lambda^{DM} \geq r}$  into  $SH_\Lambda(k)_{t_\Lambda^{SH} \geq r}$  (for any  $r \in \mathbb{Z}$ ).*

4. *Assume in addition that  $S$  contains  $p$  whenever  $p > 0$ . Then for any  $M \in \text{Obj } DM_\Lambda^c(k)$  there exists  $r \in \mathbb{Z}$  such that  $\nu_{DM_\Lambda}^{\geq r}(M) = 0$ .*

*Proof.* 1. The first half of the statement is obvious (cf. the proof of Proposition 2.2.3(2)). The second part can be proved similarly to Corollary 3.3.7(2) of [BoD15] (and follows from it in "most" cases; in the remaining cases the arguments of loc. cit. may be combined with Theorem 5.2.6 of [Mor03]).

2. We certainly have  $\overline{H}_\Lambda = \nu_{SH_\Lambda}^{\geq 1}(S_\Lambda^0)$  (see Remark 2.2.5). Thus the result is immediate from the previous assertion.

3. Similarly to the proof of Proposition 2.2.3, it suffices to "control"  $U_{k,\Lambda}(M_{gm,\Lambda}(X))$  for  $X \in SmVar$ . We certainly have  $U_{k,\Lambda}(M_{gm,\Lambda}(X)) \cong H_\Lambda \otimes \Sigma_{T,\Lambda}^\infty(X_+)$ . The previous assertion certainly yields that  $H_\Lambda \in SH_\Lambda^{eff}(k)_{t_\Lambda^{SHeff} \geq 0}$ ; thus Proposition 2.2.3(4) yields the result.

4. According to Proposition 2.2.3(9), there exist  $d, s \in \mathbb{Z}$  and some smooth projective  $P_i/k$  of dimension at most  $d$  such that  $M$  belongs to the smallest thick subcategory of  $DM_\Lambda(k)$  containing  $M_{gm,\Lambda}(P_i)\{s\}$ . Then the duality argument used in the proof of Proposition 4.25 of [Ayo15] easily yields

that we can take  $r = s + d + 1$  in our assertion. Indeed, this reasoning (for  $\Lambda$  being an arbitrary  $\mathbb{Z}[\frac{1}{p}]$ -algebra) reduces the latter claim to the vanishing of the higher Chow groups of negative codimensions for a smooth  $k$ -variety.  $\square$

### 2.3 On the "homotopy $t$ -structure conservativity" of $M_{k,\Lambda}^c$

Now we are able to prove a much stronger version of Proposition 2.1.1(I.2,II).

**Theorem 2.3.1.** *Let  $E \in \text{Obj } SH_{\Lambda}^c(k) \setminus SH_{\Lambda}(-)_{t_{\Lambda}^{SH} \geq r}$  for some  $r \in \mathbb{Z}$ . Then  $M_{k,\Lambda}(E) \notin DM_{\Lambda}(k)_{t_{\Lambda}^{DM} \geq r}$  whenever either (i)  $k$  is non-orderable or (ii)  $E$  is 2-torsion.*

*Proof.* First we assume that  $k$  is non-orderable. Then once again (by Corollary 1.4.2; cf. the proof of Proposition 2.1.1(I.2)) we can present  $k$  as  $\varinjlim k_i$ , where the cohomological (2)-dimensions of  $k_i$  are finite. Now recall that the functor  $F \mapsto DM_{\Lambda}(F)_{t_{\Lambda}^{DM} \geq r} \cap \text{Obj } DM_{\Lambda}^c(F)$  (from  $\mathcal{P}\mathcal{F}i$  into sets) is  $DM_{\Lambda}^c$ -continuous; see Proposition 2.2.3(5). Hence Remark 1.3.2(3) (combined with Proposition 2.2.3(1)) enables us to assume that the cohomological dimension of  $k$  is finite.

Now, under this additional assumption the  $\Lambda = \mathbb{Z}$ -case of our assertion is given by Theorem 15(b) of [Bach16]. In the general case we note that  $E$  may be considered as an object of  $SH(k)$  via the embedding  $G$  mentioned in Proposition 1.2.5(4);  $G(E)$  is certainly homotopy connective and slice-connective in  $SH(k)$  (see Remark 2.2.2(1)). Hence this case of our assertion follows from version (i) of loc. cit.

Lastly, in the case (ii) we argue similarly to the proof of Proposition 2.1.1(II). So we can (and will) assume that  $k$  is a finitely generated field of characteristic 0. Once again,  $E$  yields a (2-torsion) homotopy connective and slice-connective object of  $SH(k)$ . So it remains to apply Lemma 18 of ibid.  $\square$

**Corollary 2.3.2.** *If  $k$  is non-orderable then the functor  $M_{k,\Lambda}^c$  is conservative. Moreover, the restriction of  $M_{k,\Lambda}^c$  to the subcategory of 2-torsion objects is conservative for any (perfect)  $k$ .*

*Proof.* Certainly, this statement is equivalent to the  $\Lambda$ -linear version of Proposition 2.1.1(I.2,II). So, for  $E \in \text{Obj } SH_{\Lambda}^c(k)$  such that  $M_{k,\Lambda}(E) = 0$  we should check that  $E = 0$  whenever either  $k$  is non-orderable or  $E$  is 2-torsion. Now,

if  $E \neq 0$  then Proposition 2.2.3(6) yields the existence of an integer  $r$  such that  $E \notin SH_\Lambda(-)_{t_\Lambda^{SH} \geq r}$ . Hence the assertion follows from Theorem 2.3.1.  $\square$

Combining this corollary with a theorem from [Bach15], we easily obtain the following result (slightly generalizing another Bachmann's statement).

**Proposition 2.3.3.** *Assume  $p \neq 2$ ,  $k$  is non-orderable, and  $S$  contains  $p$  if  $p > 0$ . Let  $\phi$  be a non-zero quadratic form and  $a \in k \setminus 0$ . Then for the affine variety  $X$  given by the equation  $\phi = a$  the object  $C = \text{Cone}(\Sigma_{T,\Lambda}^\infty(X_+) \rightarrow S_\Lambda^0)$  (corresponding to the structure morphism for  $X$ ) is  $\otimes$ -invertible in  $SH_\Lambda(k)$ .*

*Proof.* Firstly note that  $\phi$  may be assumed to be non-degenerate. Indeed, if the kernel of (the bilinear form corresponding to)  $\phi$  is of dimension  $j \geq 0$  and  $\phi'$  is the corresponding non-degenerate form then  $X$  is isomorphic to the product of the zero set  $X'$  of  $\phi' - a$  by  $\mathbb{A}^j$ . Thus we have  $\Sigma_{T,\Lambda}^\infty(X'_+) \cong \Sigma_{T,\Lambda}^\infty(X_+)$  (see Proposition 1.3.1(2)).

Next,  $C \in \text{Obj } SH_\Lambda^c(k)$ ; hence it is dualizable (see Proposition 2.2.3(9)). Thus we should check whether the evaluation morphism  $C \otimes C^\vee \rightarrow S_\Lambda^0$  is invertible (where  $C^\vee$  is the dual to  $C$ ). Since  $M_{k,\Lambda}^c$  is symmetric monoidal and also conservative (in this case), it suffices to verify that a similar fact is valid in  $DM_\Lambda^c(k)$ . The latter is immediate from Theorem 30 of [Bach15] (where  $\phi$  was assumed to be non-degenerate).  $\square$

The following generalization of Theorem 2.2.1 of [Aso16] follows easily also.

**Proposition 2.3.4.** *Assume that  $k$  is non-orderable; let  $X$  be a smooth variety. Then the following statements are equivalent.*

1. *The morphism  $\Sigma_{T,\Lambda}^\infty(X_+) \rightarrow S_\Lambda^0$  (induced by the structure morphism  $X \rightarrow \text{Spec } k$ ) gives  $H_0^{t_\Lambda^{SH}}(\Sigma_{T,\Lambda}^\infty(X_+)) \cong H_0^{t_\Lambda^{SH}}(S_\Lambda^0)$ .*
2. *We have a similar isomorphism  $H_0^{t_\Lambda^{SH^{eff}}}(\Sigma_{T,\Lambda}^\infty(X_+)) \rightarrow H_0^{t_\Lambda^{SH^{eff}}}(S_\Lambda^0)$  (here we consider  $\Sigma_{T,\Lambda}^\infty(X_+)$  and  $S_\Lambda^0$  as objects of  $SH_\Lambda^{eff}(k)$ ).*
3. *We have  $H_0^{t_\Lambda^{DM}}(M_{gm,\Lambda}(X)) \cong H_0^{t_\Lambda^{DM}}(\Lambda)$ .*
4. *We have  $H_0^{t_\Lambda^{DM^{eff}}}(M_{gm,\Lambda}(X)) \cong H_0^{t_\Lambda^{DM^{eff}}}(\Lambda)$ .*

*Proof.* Note that  $\Sigma_{T,\Lambda}^\infty(X_+)$  and  $S_\Lambda^0$  belong to  $SH_\Lambda^{eff}(k)_{t_{SH^{eff}} \geq 0}$ , whereas  $M_{gm,\Lambda}(X)$  and  $\Lambda$  belong to  $DM_\Lambda^{eff}(k)_{t_{DM^{eff}} \geq 0}$ . Thus Proposition 2.2.6 easily yields that condition 1 is equivalent to condition 2, and 3 is equivalent to 4. Next, condition 1 implies condition 3 by Proposition 2.2.3(3).

Lastly, assume that  $H_0^{t_{DM}^\Lambda}(M_{gm,\Lambda}(X)) \cong H_0^{t_{DM}^\Lambda}(\Lambda)$ . Applying Theorem 2.3.1 (version (i)) in the case  $r = 1$ ,  $E = \text{Cone}(\Sigma_{T,\Lambda}^\infty(X_+) \rightarrow S_\Lambda^0)$ , we obtain that  $E \in SH_\Lambda(k)_{t_{SH} \geq 1}$ . Applying Proposition 2.2.3(10) (and considering the exact sequence  $\cdots \rightarrow SH_\Lambda(k)(S_\Lambda^0, \Sigma_{T,\Lambda}^\infty(X_+) \rightarrow SH_\Lambda(k)(S_\Lambda^0, S_\Lambda^0) \rightarrow SH_\Lambda(k)(S_\Lambda^0, E) = \{0\}$ ) we obtain a splitting  $\Sigma_{T,\Lambda}^\infty(X_+) \cong S_\Lambda^0 \oplus E[1]$ . Thus the application of Theorem 2.3.1 to our  $E$  (in the case  $r = 1$ ) also yields that our condition 3 implies condition 1.  $\square$

*Remark 2.3.5.* 1. Surely, for  $E$  as above one can also apply Theorem 2.3.1 to the spectrum  $E[2] = \text{Cone}(E \xrightarrow{\times 2} E)$  (without having to assume that  $k$  is non-orderable).

2. Now assume in addition that  $X$  is (also) proper.

Then one can easily see (cf. Lemma 2.1.3 of [Aso16]; here  $k$  may be any perfect field) that condition 4 (and 3) of our Proposition is fulfilled if and only if the kernel of the degree homomorphism  $\text{Chow}_0(X_L) \rightarrow \mathbb{Z}$  is  $S$ -torsion and  $X_L$  contains a zero-cycle whose degree belongs to  $S$  for any field extension  $L/k$ . Certainly, it suffices to verify the latter condition for  $L = k$  only.

3. Under the assumption that  $p$  belongs to  $S$  whenever it is positive one may formulate a much more general result of this sort. Indeed, Corollary 3.3.2 of [BoS14] gives for  $M \in \text{Obj } DM_\Lambda^{eff,c}(k)$  several conditions equivalent to  $M \in DM_\Lambda^{eff}(k)_{t_{DM^{eff}} \geq 0}$  (note that in *ibid.* the cohomological convention for  $t$ -structures is used). Most of these conditions are formulated in terms of the so-called Chow-weight homology of  $M$ . Thus assuming that  $E \in \text{Obj } SH_\Lambda^{eff,c}(k)$  is 2-torsion whenever  $k$  is formally real, one obtains an answer to the question whether  $M_{k,\Lambda}(E)$  belongs to  $SH_\Lambda^{eff}(k)_{t_{SH^{eff}} \geq 0}$  in terms of certain complexes of Chow groups corresponding to  $E$ .

4. Certainly, in the case  $\Lambda = \mathbb{Z}$  and  $X$  being proper we could have deduced (most of) our proposition directly from Theorem 2.2.1 of [Aso16] (using the  $DM^c$ -continuity of  $DM(-)_{t_{hom}^{DM} \geq 0} \cap \text{Obj } DM^c(-)$ ).



### 3 The relation to infinitely effective spectra and cobordism-modules

This section is dedicated to those results on the motivization kernel that are not (closely) related to the ones in the literature.

In §3.1 we prove that the compact motivization functor  $M_{k,\Lambda}^c$  "strictly respects" the slice filtrations (on  $SH_\Lambda^c(k)$  and  $DM_\Lambda^c(k)$ , respectively); it also "detects" the filtration  $\text{Fil}_{\text{Tate}}^*$  (see Definition 2.2.4) on the lower  $t_{\text{hom}}$ -homology of an object of  $SH_\Lambda^c(k)$ .

In §3.2 we describe an alternative method of the proof of Theorem 2.3.1(i) (under the additional assumption that  $p$  is invertible in  $\Lambda$  whenever it is positive).

In §3.3 we explain that in all our results the categories  $DM_\Lambda(-)$  may be replaced by the ( $\Lambda$ -linearized) categories  $D_\Lambda^{\text{MGl}}(-)$  of (strict) modules over the Voevodsky's motivic cobordism spectrum  $\text{MGl}$ .

#### 3.1 The effectivity description of the "compact motivization kernel"

Now we prove "the most original" result of this paper.

**Theorem 3.1.1.** *Let  $r, m \in \mathbb{Z}$  and let  $E$  be slice-connective.*

*Then the following statements are valid.*

I1.  *$E \in SH_\Lambda^{\text{eff}}(k)\{r\}$  if and only if  $M_{k,\Lambda}(E) \in DM_\Lambda^{\text{eff}}(k)\{r\}$ . In particular,  $E \in \bigcap_{j \in \mathbb{Z}} SH_\Lambda^{\text{eff}}(k)\{j\}$  (i.e., it is infinitely effective) if and only if  $M_{k,\Lambda}(E)$  also is.*

2. *Assume that  $E \in SH_\Lambda(k)_{t_\Lambda^{\text{SH}} \geq m}$ . Then  $\text{Fil}_{\text{Tate}}^r H_m^{t_\Lambda^{\text{SH}}}(E)$  (see Definition 2.2.4(3)) equals  $H_m^{t_\Lambda^{\text{SH}}}(E)$  if and only if  $\text{Fil}_{\text{Tate}}^r H_m^{t_\Lambda^{\text{DM}}}(M_{k,\Lambda}(E)) = H_m^{t_\Lambda^{\text{DM}}}(M_{k,\Lambda}(E))$ .*

II *Assume in addition that  $E \in \text{Obj } SH_\Lambda^c(k)$ ; if  $p > 0$  then suppose also that  $p \in S$ .*

1.  *$E$  is infinitely effective whenever  $M_{k,\Lambda}(E) = 0$ .*

2. *If  $E \in SH_\Lambda(k)_{t_\Lambda^{\text{SH}} \geq m}$  then  $M_{k,\Lambda}(E) \in DM_\Lambda(k)_{t_\Lambda^{\text{DM}} \geq m+1}$  if and only if for any  $s \in \mathbb{Z}$  we have  $\text{Fil}_{\text{Tate}}^s H_m^{t_\Lambda^{\text{SH}}}(E) = H_m^{t_\Lambda^{\text{SH}}}(E)$ .*

*Proof.* I.1. Proposition 2.2.3(3) immediately gives the "only if" implication.

Now we prove the "if" part of the assertion; so we assume that  $M_{k,\Lambda}(E) \in \text{Obj } DM_\Lambda^{\text{eff}}(k)\{r\}$ . By the definition of slice-effectivity,  $E$  belongs to  $\text{Obj } SH_\Lambda^{\text{eff}}(k)\{r'\}$

for some  $r' \in \mathbb{Z}$ ; we take the maximal  $r' \leq r$  that fulfils this condition. Consider the distinguished triangle

$$E \rightarrow H_\Lambda \otimes E = U_{k,\Lambda}(M_{k,\Lambda}) \rightarrow \overline{H}_\Lambda \otimes E \quad (1)$$

(see Proposition 2.2.3(7)). Recall that  $\overline{H}_\Lambda \in SH_\Lambda^{eff}(k)\{1\}$ ; hence  $\overline{H}_\Lambda \otimes E \in SH_\Lambda^{eff}(k)\{r'+1\}$  (see part 4 of the proposition). Therefore  $E \in \text{Obj } SH_\Lambda^{eff}(k)\{r'+1\}$  whenever  $r' < r$ . Thus  $r = r'$ .

2. First we verify the "only if" part of the assertion. If the morphism  $a_r(E) : \nu_{SH_\Lambda}^{\geq r}(E) \rightarrow E$  yields a surjection on  $H_m^{t^{SH}}(-)$  then the right  $t$ -exactness of  $M_{k,\Lambda}$  implies that the morphism  $M_{k,\Lambda}(a_r(E)) : M_{k,\Lambda}(\nu_{SH_\Lambda}^{\geq r}(E)) \rightarrow M_{k,\Lambda}(E)$  gives a surjection on  $H_m^{t^{DM}}(-)$ . It remains to note that  $M_{k,\Lambda}(\nu_{SH_\Lambda}^{\geq r}(E)) \in DM_\Lambda^{eff}(k)\{r\}$  (by Proposition 2.2.3(3)); hence  $M_{k,\Lambda}(a_r(E))$  factors through  $\nu_{DM_\Lambda}^{\geq r}(M_{k,\Lambda}(E))$  (see Remark 2.2.5).

Conversely, assume that the morphism  $b_r(E) : \nu_{DM_\Lambda}^{\geq r}(M_{k,\Lambda}(E)) \rightarrow M_{k,\Lambda}(E)$  yields a surjection on  $H_m^{t^{DM}}(-)$ . Then an argument similar to the one we have just used (and also relying on Remark 2.2.5) yields that the morphism  $\nu_{SH_\Lambda}^{\geq r}(H_\Lambda \otimes E) \rightarrow H_\Lambda \otimes E$  induces a surjection on  $H_m^{t^{SH}}(-)$ . Besides, there certainly exists  $r' \in \mathbb{Z}$  such that  $\text{Fil}_{\text{Tate}}^{r'} H_m^{t^{SH}}(E) = H_m^{t^{SH}}(E)$ , and we choose the maximal  $r' \leq r$  that fulfils this condition. Now we consider the diagram

$$\begin{array}{ccccc} H_{m+1}^{t^{SH}}(\nu_{SH_\Lambda}^{\geq r'+1}(\overline{H}_\Lambda \otimes E)) & \longrightarrow & H_m^{t^{SH}}(\nu_{\geq r'+1}^{SH}(E)) & \xrightarrow{b} & H_m^{t^{SH}}(\nu_{SH_\Lambda}^{\geq r'+1}(H_\Lambda \otimes E)) \\ \downarrow c & & \downarrow d & & \downarrow e \\ H_{m+1}^{t^{SH}}(\overline{H}_\Lambda \otimes E) & \longrightarrow & H_m^{t^{SH}}(E) & \xrightarrow{g} & H_m^{t^{SH}}(H_\Lambda \otimes E) \end{array}$$

Certainly, the rows are exact (in the middle). Moreover, the morphisms  $b$  and  $g$  are surjective by Proposition 2.2.6(1,2) (along with Proposition 2.2.3(4)).

Next, we certainly have a commutative triangle  $H_{m+1}^{t^{SH}}(\overline{H}_\Lambda \otimes \nu_{SH_\Lambda}^{\geq r'}(E)) \rightarrow H_{m+1}^{t^{SH}}(\nu_{SH_\Lambda}^{\geq r'+1}(\overline{H}_\Lambda \otimes E)) \rightarrow H_{m+1}^{t^{SH}}(\overline{H}_\Lambda \otimes E)$  (by Proposition 2.2.6 combined with Remark 2.2.5); thus  $c$  is surjective. Suppose that  $r' < r$ ; then  $e$  is surjective also. Thus  $d$  is surjective in this case, and we obtain a contradiction. Hence  $r' = r$ .

II.1. If  $M_{k,\Lambda}(E) = 0$  then  $E$  is infinitely effective according to assertion I.1.

Conversely, since  $M_{k,\Lambda}(\text{Obj } SH_\Lambda^{eff}(k)\{s\}) \subset \text{Obj } DM_\Lambda^{eff}(k)\{s\}$  for any  $s \in \mathbb{Z}$ , we obtain that  $M_{k,\Lambda}$  respects infinite effectivity. Lastly, Proposition

2.2.3(8) yields that there are only zero infinitely effective compact objects in  $DM_\Lambda$ .

2. According to assertion I.2, for any  $s \in \mathbb{Z}$  we have  $\text{Fil}_{\text{Tate}}^s H_m^{t^{SH}}(E) = H_m^{t^{SH}}(E)$  if and only if  $\text{Fil}_{\text{Tate}}^s H_m^{t^{DM}}(M_{k,\Lambda}(E)) = H_m^{t^{DM}}(M_{k,\Lambda}(E))$ . Now, by Proposition 2.2.6(4), the latter is equivalent to  $H_m^{t^{DM}}(M_{k,\Lambda}(E)) = 0$  (if we take  $s$  being large enough); combined with Proposition 2.2.3(3) this yields the result.  $\square$

*Remark 3.1.2.* So we obtain that  $M_{k,\Lambda}^c$  induces an exact conservative functor from the localization of  $SH_\Lambda^c(k)$  by its subcategory of infinitely effective objects into  $DM_\Lambda^c(k)$  (under the condition that  $p \in S$ ).

Now, recall that the same restriction on  $S$  ensures the existence of an exact conservative *weight complex* functor  $DM_\Lambda^c(k) \rightarrow K^b(\text{Chow}_\Lambda(k))$  (that was essentially constructed in [Bon10] and in [Bon11] in the case  $p > 0$ ; see [BoI15, Propositions 3.1.1, 2.3.2] for the  $\Lambda$ -linear formulation). So the composition functor is conservative also; if  $k$  is non-orderable this is actually a functor  $SH_\Lambda^c(k) \rightarrow K^b(\text{Chow}_\Lambda(k))$  (by Corollary 2.3.2). Note however that  $\eta \neq 0$  unless  $2 \in S$  (by Theorem 6.3.3 of [Mor03]); hence there cannot exist a Chow weight structure for  $SH_\Lambda^c(k)$  in this case (easy; see Remark 6.3.1(3) of [Bon13]) and this composition version of the weight complex functor does not come from a weight structure.

### 3.2 An alternative proof of Theorem 2.3.1(i) (with $p$ inverted)

Now we describe an alternative proof of version (i) of Theorem 2.3.1 that relies on [Lev13] instead of [Bach16] (that is actually based on certain results of M. Levine also). This reasoning requires us to assume that  $p \in S$  whenever  $p > 0$ . Note also that this version of our theorem certainly implies the  $\mathbb{Z}[\frac{1}{p}]$ -linear version of Proposition 2.1.1(I.2).

So, we suppose that  $k$  is non-orderable. Then the continuity argument used in the proof of Theorem 2.3.1 allows us to assume (once again) that the cohomological dimension of  $k$  is finite.

According to Theorem 3.1.1(II.2) (combined with Remark 2.2.2 and Proposition 2.2.3(3)), it suffices to prove (under our assumption on  $S$  and for some  $m \in \mathbb{Z}$ ) that the filtration  $\text{Fil}_{\text{Tate}}^*$  on  $H_m^{t^{SH}}(E)$  is non-trivial (i.e., that

$H_m^{t^{SH}}(E)$  does not lie in its own  $\text{Fil}_{\text{Tate}}^s$  for all  $s \in \mathbb{Z}$  for any  $E$  belonging to  $\text{Obj } SH_{\Lambda}^c(k) \cap SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq m} \setminus SH_{\Lambda}(k)_{t_{\Lambda}^{SH} \geq m+1}$ .

Next, the very well-known Proposition 6.1.1(4) of [Bon13] yields the following: it suffices to verify that the filtration in question is "separated at stalks", i.e., that for any finitely generated field  $L/k$  and  $j \in \mathbb{Z}$  the filtration induced by  $\text{Fil}_{\text{Tate}}^* H_m^{t^{SH}}(E)$  on the result of the "evaluation of  $E\{j\}$  at  $L$ " (see §3.2.1 of [Deg13]) is separated.

Once again, we consider  $E$  as an object of  $SH$  using the embedding  $G$  described in Proposition 1.2.5(4). Then it is cohomologically finite in the sense of Definition 6.1 of [Lev13]. Indeed,  $E$  satisfies condition (i) of loc. cit. by Remark 2.2.2. It satisfies condition (ii) of the definition according to Proposition 6.9(3) of ibid. combined with Proposition 2.2.3(9) above. Hence the separatedness in question is given by Theorem 7.3 of [Lev13]; this finishes the proof.

*Remark 3.2.1.* 1. The "yoga" of this argument (as well as of Theorem 2.3.1 itself) is that certain types of assertions concerning **compact** objects of  $SH_{\Lambda}$  can be reduced to the case where the (virtual) cohomological dimension of  $k$  is finite. Now, under this additional assumption one can apply the appropriate properties (as studied Bachmann and Levine) of certain subcategories of  $SH(k)$  that are bigger than  $SH^c(k)$  (or  $SH_{\Lambda}^c(k)$  for the corresponding  $\Lambda$ ). So, in our main statements we restrict ourselves to compact motivic spectra; this enables us to establish them over a wide class of base fields. This method appears to be quite useful since (most of) motivic spectra "coming from geometry" are compact. Also, one "usually" does not apply  $M_k$  to non-compact objects of  $SH$  (that are mostly used for representing various cohomology theories).

2. One may also apply some of the arguments of [Lev13] for proving version (ii) of Theorem 2.3.1.

### 3.3 On the cobordism-module versions of the main results

Now recall that the category  $DM(k)$  is closely related to the homotopy of highly structured modules over the ring object  $H_{\mathbb{Z}}$  in the model category of motivic symmetric spectra underlying  $SH$  (see Proposition 38 of [OsR08]).

The goal of this section is to explain that our main results are also valid if we replace  $M_k$  (and  $M_{k,\Lambda}$ ) by the corresponding functor  $M_k^{\text{MGl}} : SH(k) \rightarrow$

$D^{\text{MGl}}(k)$ , where the latter is the (stable) homotopy category of the category  $\text{MGl} - \text{Mod}$  of (strict left) modules over the Voevodsky's spectrum  $\text{MGl}$  (see §1.3 of [BoD15]). Similarly to [OsR08], one can verify the existence of  $M_k^{\text{MGl}}$  given by the "free  $\text{MGl}$ -module functor"; the corresponding forgetful functor yields the right adjoint  $U_k^{\text{MGl}}$  to  $M_k^{\text{MGl}}$ . For  $S \subset \mathbb{P}$  we will also consider the corresponding  $D_\Lambda^{\text{MGl}}(-)$ .

Now assume that  $S$  contains  $p$  if  $p > 0$  (the author is not sure whether this is really necessary).

Then there are three possible ways of proving Theorem 2.3.1 with  $M_{k,\Lambda}$  replaced by  $M_{k,\Lambda}^{\text{MGl}}$  (and for the corresponding homotopy  $t$ -structure for  $D_\Lambda^{\text{MGl}}(k)$ ).

Firstly, one may prove that in all the results of §2.2 one may replace  $DM_\Lambda(-)$  by  $D_\Lambda^{\text{MGl}}(-)$ . The key points here are the following ones:  $\text{MGl} \in SH(k)_{t_{hom}^{SH} \geq 0}$  (see Corollary 3.9 of [Hoy15]); the corresponding analogue of Proposition 2.2.3(9) is given by Theorem 5.2.6 of [Mor03], whereas the  $\text{MGl}$ -analogue of Proposition 2.2.6(4) follows from the vanishing of  $SH(F)(S^0, \text{MGl}(j)[i])$  for any  $i \in \mathbb{Z}$ ,  $j < 0$ , and any perfect field  $F$  (that follows from Theorem 8.5 of [Hoy15]). Having these statements one can easily verify that the corresponding analogue of Theorem 3.1.1 is valid also (probably one can also deduce this statement from Lemma 7.10 of *ibid.*). The latter allows to deduce the result in question from its  $DM_\Lambda(-)$ -analogue (i.e., from Theorem 2.3.1) immediately.

Another possibility is to deduce the  $D_\Lambda^{\text{MGl}}$ -analogue of Theorem 2.3.1 from the corresponding version of Theorem 11 of [Bach16]; the latter statement easily follows from Remark 4.3.3 of [BoD15].

Lastly, one may apply the important diagram

$$\begin{array}{ccc} SH_\Lambda(k) & \xrightarrow{M_{k,\Lambda}^{\text{MGl}}} & D_\Lambda^{\text{MGl}}(k) \\ \downarrow & & \downarrow \\ D_{\mathbb{A}^1, \Lambda}(k) & \xrightarrow{M_{k,\Lambda}^D} & DM_\Lambda(k) \end{array}$$

(certainly, along with its "compact" analogue) to the study of the conservativity of connecting functors. Here we use the notation of (Example 1.3.3 of) [BoD15] for the lower left hand corner of this diagram; cf. also §1 of [Bach16].

This diagram (along with Proposition 2.3.7 of *ibid.*) also demonstrates that one may replace all  $M_{-, \Lambda}$  in our statements by  $M_{-, \Lambda}^D$ .

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